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Quantum speed limits in open system dynamics

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Bounds to the speed of evolution of a quantum system are of fundamental interest in quantum metrology, quantum chemical dynamics and quantum computation. We derive a time-energy uncertainty relation for open quantum systems undergoing a general, completely positive and trace preserving (CPT) evolution which provides a bound to the quantum speed limit. When the evolution is of the Lindblad form, the bound is analogous to the Mandelstam-Tamm relation which applies in the unitary case, with the role of the Hamiltonian being played by the adjoint of the generator of the dynamical semigroup. The utility of the new bound is exemplified in different scenarios, ranging from the estimation of the passage time to the determination of precision limits for quantum metrology in the presence of dephasing noise.

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How fast can a quantum system evolve? Quantum mechanics acts as a legislative body imposing speed limits to the evolution of quantum systems. While these limits are both ultimate and fundamental, at the same time, their existence is at the center of a surge of activity, as a result of their manifold applications, including the identification of precision bounds in quantum metrology [1], the formulation of computational limits of physical systems [2], and the development of quantum optimal control algorithms [3].

Bounds on the speed of evolution are intimately related to the concept of the passage time τ_P, which is the required time for a given pure state $|\chi\rangle$ to become orthogonal to itself under unitary dynamics [4]. One of the early answers to this problem was provided by Mandelstam and Tamm (MT), who showed that the passage time can be lower-bounded by the inverse of the variance in the energy of the system so that

$$\tau \geq \frac{\pi}{2}\frac{\hbar}{\Delta H}, \tag{1}$$

where $\Delta H = (\langle H^2\rangle - \langle H\rangle^2)^{1/2}$, whenever the dynamics under study is governed by an Hermitian Hamiltonian H [5–8]. A simple geometric interpretation of this result was provided by Brody using the Fubini-Study metric in the Hilbert space spanned by the initial state and its orthogonal complement [9]. Indeed, the passage time problem can be posed as a quantum brachistochrone problem. From this perspective, a particularly exciting result was found: whenever the Hamiltonian is non-Hermitian PT-symmetric, the passage time can be made arbitrarily small without violating the time-energy uncertainty principle [10, 11]. A second bound, due to Margolus and Levitin (ML), takes the simpler form $\tau \geq \frac{\pi}{2}\frac{\hbar}{\langle H\rangle - E_0}$ where the zero of energy is generally shifted to the ground state energy so that $E_0 = 0$ [12]. This bound has been applied to ascertain fundamental computational limits in nature [2, 13].

Despite the growing body of literature on the subject, the analysis has almost exclusively been focused on unitary dynamics of isolated quantum systems. An analogous bound for open quantum systems is highly desirable, since ultimately all systems are coupled to an environment [14, 15]. As an example, such a bound on the evolution of an open system would help to address the robustness of quantum simulators and computers against decoherence [16], as well as the relevance of the specific nature of the noise, and in particular whether or not it is Markovian, in phase estimation problems of interest in metrology and precision spectroscopy [17, 18].

The MT bound can be derived by considering the time evolution of the overlap $\alpha = |\langle\psi_t|\psi_0\rangle|$ between the initial state $|\psi_0\rangle$ and the quantum state $|\psi_t\rangle$ at time t subject to a unitary evolution $U(t) = \exp\{-iHt/\hbar\}$. It can be shown that the MT-limit (eq. 1) is achievable, as for a suitable Hamiltonian H we can satisfy the differential equation $\hbar\frac{d\alpha^2}{dt} = -2\Delta H\alpha\sqrt{1-\alpha^2}$ which for $\alpha = \cos\phi$ is easily seen to result in $\hbar\dot{\phi} = \Delta H$ thus matching the MT bound [19].

In the case of open system dynamics we need to consider general non-unitary quantum evolutions and have the freedom to choose a variety of distance measures between quantum states. One natural choice here is the fidelity between two mixed states ρ and σ, which is given by $F(\rho,\sigma) = \mathrm{tr}[\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}]$. The quantum speed limit then provides a lower bound on the time τ that is required to achieve, for a given initial state $\rho(0)$ and a target fidelity f_{target}, the condition $F(\rho_t,\rho_0) < f_{target}$ subject to an open system evolution. Ideally, such bounds should reduce to the MT bound in the case of unitary dynamics on pure states and/or be easy to compute.

Bounds on τ may be derived by taking inspiration from the variational characterization of the fidelity $F(\rho^S,\sigma^S) = \max[|\langle\psi^{SE}|\phi^{SE}\rangle|]$ [20], where the maximization is over all $|\psi^{SE}\rangle$ ($|\phi^{SE}\rangle$) on a larger Hilbert space $\mathscr{H}^{SE}$ that are purifications of the mixed states ρ^S (σ^S) on the smaller system S, that is $\mathrm{tr}_E[|\psi^{SE}\rangle\langle\psi^{SE}|] = \rho^S$ ($\mathrm{tr}_E[|\phi^{SE}\rangle\langle\phi^{SE}|] = \sigma^S$). Then for any specific purification the inequality $F(\rho^S,\sigma^S) \geq |\langle\psi^{SE}|\phi^{SE}\rangle|$ holds. A general time evolution of a subsystem $\dot{\rho}^S = \mathscr{L}\rho^S$

can always be generated by a joint unitary dynamics U_t^{SE} of the system with an environment such that

$$\rho_t^S = \mathrm{tr}_B[U_t^{SE}|\psi_0^{SE}\rangle\langle\psi_0^{SE}|U_t^{\dagger,SE}]$$

where $\rho_0^S = \mathrm{tr}_B[|\psi_0^{SE}\rangle\langle\psi_0^{SE}|]$. Such a dynamics will be generated by a suitable Hamiltonian H_t^{SE} but it should be noted that the choice of $|\psi_0^{SE}\rangle$, U_t^{SE} and thus H_t^{SE} is not unique.

Now we can make use of the fact that

$$f_{target} \geq F(\rho_t^S, \rho_0^S) \geq F(U_t^{SE}|\psi_0^{SE}\rangle, |\psi_0^{SE}\rangle)$$

for any choice of purification of ρ_t^S and ρ_0^S and any choice of unitary dynamics U_t^{SE} that generates $\dot{\rho}^S = \mathscr{L}\rho^S$ on the subsystem. This implies that any choice of purification and unitary evolution will achieve $f_{target} \geq F(U_t^{SE}|\psi_0^{SE}\rangle, |\psi_0^{SE}\rangle)$ at an earlier time t than $f_{target} \geq F(\rho_\tau^S, \rho_0^S)$, i.e. $t < \tau$. As a consequence, for any choice of $|\psi_0^{SE}\rangle$, U_t^{SE} and thus H_t^{SE} we obtain a lower bound on τ. If H_t^{SE} is given, then we can compute ΔH_t^{SE} in the state $|\psi_t^{SE}\rangle$ and immediately provide a lower bound on τ via the MT and $\hbar\Delta\phi = \hbar\int_0^t ds\dot{\phi} \leq \int_0^t ds \Delta H_s^{SE}$. Needless to say, performing the optimization over all possible purifications and all possible H_t^{SE} is a challenging task that will be very hard to perform in the general case. Two routes are suggested themselves. Firstly, well chosen $|\psi_0^{SE}\rangle$, U_t^{SE} and thus H_t^{SE} will lead to excellent bounds for reasonably simple cases. Secondly, analytical lower bounds on τ may also be obtained by studying different distance measures that are easier to handle and thus admit closed formulae for lower bounds.

Here we follow this second approach to find an analytical and easy to compute lower bound on the speed of evolution in open quantum systems. We shall derive a bound analogous to the seminal result by MT where the energy variance of the initial state is replaced by a more general measure taking into account the coupling to the environment. We shall pay particular attention to the dynamics governed by a dynamical semigroup in which case the evolution of the system is ruled by a master equation of the Lindblad form [21]. It has recently been pointed out that the MT and ML bounds are violated under non-unitary dynamics [22]. We shall nonetheless show in the following that Markovian systems are subjected to a MT-type of bound where the adjoint of the generator of the dynamical semigroup plays the role of the system Hamiltonian in the unitary case.

Decay of an open quantum system Consider a given system described by a state ρ_0 (from now we drop the upper index S for convenience) coupled to an environment in a state ρ_0^E, and assume both system and environment are weakly coupled such that the initial global state can be approximated by $\rho_0 \otimes \rho_0^E$. Let the global reversible dynamics be governed by a unitary evolution operator U_t. The reduced dynamics of the system is given by a one-parameter family of dynamical maps $\rho \mapsto \mathscr{V}_t\rho := \mathrm{tr}_E[U_t\rho_0 \otimes \rho_0^E U_t^\dagger]$, parameterized by the time variable $t \in \mathbb{R}^+$. Whenever the typical time scale of the environment is much smaller than that of the system, one can assume a Markovian dynamics. Under Markovian dynamics, such maps form a quantum dynamical semigroup $\mathscr{V}_{t+s}\rho = \mathscr{V}_t\mathscr{V}_s\rho$, $t,s > 0$ (we assume that the open system is not subjected to an external time-dependent field so that the generator of the quantum dynamical semigroup is time independent). Any such map can be represented by a Markovian master equation

$$\frac{d\rho_t}{dt} = \mathscr{L}\rho_t, \tag{2}$$

where the generator of V_t admits the Lindblad form [21]

$$\mathscr{L}\rho = -\frac{i}{\hbar}[H,\rho] + \sum_k F_k\rho F_k^\dagger - \frac{1}{2}\left\{F_k^\dagger F_k, \rho\right\}, \tag{3}$$

such that $\mathscr{V}_t\rho_0 = e^{t\mathscr{L}}\rho_0$. In such scenario we might pose the following question: Which is the bound to the speed of evolution from an initial state ρ_0 under the action of a quantum dynamical semigroup $\mathscr{V}_t$? To answer this question we introduce as a figure of merit the so called relative purity [23]

$$f(t) = \frac{\mathrm{tr}\left[\rho_0\rho_t\right]}{\mathrm{tr}(\rho_0^2)}, \tag{4}$$

which is a generalization of the survival probability $\mathscr{S}(t) = |\langle\chi|e^{-iHt/\hbar}|\chi\rangle|^2$ often used for a pure state $|\chi\rangle$ subject to a Hamiltonian H, and that has proved useful in studying quantum speed limits in the unitary case [13].

Derivation of the bound from the (Lindblad) master equation Let us now characterize the decay rate of the relative purity. Note that whenever the generator admits a Lindblad form (i.e. for a Markovian quantum master equation),

$$\dot{f}(t) = \frac{\mathrm{tr}\left[\rho_0\,\mathscr{L}\rho_t\right]}{\mathrm{tr}(\rho_0^2)} = \frac{\mathrm{tr}\left[\mathscr{L}^\dagger\rho_0\,\rho_t\right]}{\mathrm{tr}(\rho_0^2)} \tag{5}$$

where the adjoint of the generator of the dynamical map reads

$$\mathscr{L}^\dagger\rho_0 = \frac{i}{\hbar}[H,\rho_0] + \sum_k F_k^\dagger\rho_0 F_k - \frac{1}{2}\left\{F_k^\dagger F_k, \rho_0\right\}. \tag{6}$$

The rate of change of f can then be bounded using the Cauchy-Schwarz inequality for operators, $|\mathrm{tr}(A^\dagger B)|^2 \leq \mathrm{tr}(A^\dagger A)\mathrm{tr}(B^\dagger B)$. Then

$$\begin{aligned} |\dot{f}(t)| &\leq \sqrt{\mathrm{tr}[(\mathscr{L}^\dagger\rho_0)^2]\mathrm{tr}[\rho_t^2]}/\mathrm{tr}\rho_0^2 \\ &\leq \sqrt{\mathrm{tr}[(\mathscr{L}^\dagger\rho_0)^2]}/\mathrm{tr}\rho_0^2, \end{aligned} \tag{7}$$

that is, by making reference exclusively to the initial state and the dynamical map. Let us parametrize $f(t) = \cos\vartheta$ with $\vartheta \in [0,\pi/2]$. Upon integration between $\vartheta = 0$ ($f(0) = 1$) and a final $\vartheta = \theta$, the following bound to the required time of evolution is found

$$\tau_\theta \geq \frac{|\cos\theta - 1|\mathrm{tr}\rho_0^2}{\sqrt{\mathrm{tr}[(\mathscr{L}^\dagger\rho_0)^2]}} \geq \frac{4\theta^2\mathrm{tr}\rho_0^2}{\pi^2\sqrt{\mathrm{tr}[(\mathscr{L}^\dagger\rho_0)^2]}}. \tag{8}$$

This generalizes the MT uncertainty relation for open quantum systems governed by a Markovian quantum master equation. The generalization to a time-dependent Lindbladian

$\mathscr{L}(t)$ is straightforward and reads

$$\tau_\theta \geq \frac{4\theta^2 \mathrm{tr}\rho_0^2}{\pi^2 \overline{\sqrt{\mathrm{tr}[(\mathscr{L}^\dagger \rho_0)^2]}}}. \tag{9}$$

where $\overline{X} = \tau_\theta^{-1} \int_0^{\tau_\theta} X dt$.

Derivation of the bound using general quantum channels To remove the Markovian approximation, we note that any kind of time evolution of a quantum state ρ_0 can be written in the form $\rho_t = \sum_\alpha K_\alpha(t,0)\rho_0 K_\alpha^\dagger(t,0)$. In particular, $K_\alpha(t,0)$ is independent of ρ_0 if the dynamical map is induced from an extended system with the initial condition $\rho_0^{SE} = \rho_0 \otimes \rho^E(0)$. Then, the dynamical map is said to be universal. Let such map govern the evolution and consider

$$f(t) = \mathrm{tr}[\rho_0\rho_t] = \sum_\alpha \mathrm{tr}[\rho_0 K_\alpha(t,0)\rho_0 K_\alpha^\dagger(t,0)]. \tag{10}$$

Parametrizing $f(t) = \cos\theta$, a bound can be derived

$$\tau_\theta \;\geq\; \frac{2\theta^2}{\pi^2} \frac{\sqrt{\mathrm{tr}[\rho_0^2]}}{\overline{\sum_\alpha ||K_\alpha(t,0)\rho_0 \dot{K}_\alpha^\dagger(t,0)||}} \tag{11}$$

where $||A|| = \sqrt{\mathrm{tr}(A^\dagger A)}$ is the the the Hilbert-Schmidt norm of A. Details of the derivation are provided as an Appendix.

Applications The bound to the speed of evolution presented above is the main result of this paper. In the following we shall analyze some particular cases to illustrate its use:

Passage time. - Under unitary time evolution, the passage time is the minimum time required for a time evolving state $|\chi(t)\rangle$ to become orthogonal to its initial value $|\chi(0)\rangle$. Let us consider a pure state such that $\mathrm{tr}\rho_0^2 = 1$ and let $\mathscr{L}^\dagger\rho = -\mathscr{L}\rho = i[H, |\chi\rangle\langle\chi|]/\hbar$. It follows from Eq. (9), that

$$\tau_\theta \geq \frac{\hbar}{\sqrt{2}\Delta E}. \tag{12}$$

Alternatively, for $\alpha = 1$, $K = \exp[-i(H - \langle\chi|H|\chi\rangle)t/\hbar]$, $\tau_\theta \geq \frac{\hbar}{2\Delta E}$, a factor $1/\sqrt{2}$ smaller. The usual definition of the passage time $\tau_P = \frac{\pi\hbar}{2\Delta E}$, refers to the orthogonalization measured by the square root of the fidelity (sometimes referred to as the integrity or survival probability amplitude), which for a pure states under unitary dynamics reduces to $|\langle\chi(0)|\chi(t)\rangle|$ [4–7].

Non-Hermitian Hamiltonians. - Non-Hermitian Hamiltonians are ubiquitous in quantum physics and enjoy of a wide range of applications from quantum optics [24] to reactive scattering [25]. Their standard derivation is based on Feshbach's partitioning theory, that allows to describe the effective dynamics of a quantum system governed by a Hamiltonian H, when restricted to a given subspace associated with projector P (with complement Q, such that $P + Q = 1$, $P^2 = P$, $Q^2 = Q$). The effective Hamiltonian governing the dynamics in the restricted subspace, $H_{\mathrm{eff}} = PHP + PHQ(E - QHQ)^{-1}QHP$, is generally non-Hermitian. Under H_P the density matrix $i\dot{\rho} = (H_{\mathrm{eff}}\rho - \rho H_{\mathrm{eff}}^\dagger)/\hbar$. Similarly, in open systems under Markovian dynamics it is customary to split the generator of the dynamical map in two contributions $\mathscr{L}_c$ and $\mathscr{D}$, i.e. $\mathscr{L} = \mathscr{L}_c + \mathscr{D}$. $\mathscr{L}_c$ describes the coherent evolution associated with the non-Hermitian Hamiltonian $H_{\mathrm{eff}} = H - i\hbar\frac{1}{2}\sum_k F_k^\dagger F_k$, while the dissipator $\mathscr{D}\rho = \sum_k F_k\rho F_k^\dagger$ is associated with spontaneous decay, and it is a jump operator [24]. More generally, let $H_{\mathrm{eff}} = H - i\Gamma$, where H and Γ are both Hermitian operators, so that $\mathscr{L}_c\rho = -i[H,\rho]/\hbar - \{\Gamma,\rho\}/\hbar$. Noting that upon setting $\mathscr{D}\rho = 0$, the bound to the speed of evolution under non-Hermitian Hamiltonians still holds, it follows from Eq. (7) that

$$\begin{aligned}
\tau_\theta \;&\geq\; \frac{4\theta^2\hbar\mathrm{tr}\rho_0^2}{\pi^2 \overline{\sqrt{\mathrm{tr}[(\mathscr{L}_c^\dagger\rho)^2]}}}, \\
&=\; \frac{4\theta^2\hbar\mathrm{tr}\rho_0^2}{\pi^2 \overline{\sqrt{\mathrm{tr}(-[H,\rho]^2 + \{\Gamma,\rho\}^2 - 2i[H,\Gamma]\rho^2)}}}, \\
&=\; \frac{4\theta^2\hbar}{\sqrt{2}\pi^2 \overline{\sqrt{\Delta H^2 + (\langle\Gamma^2\rangle + \langle\Gamma\rangle^2) - i\langle[H,\Gamma]\rangle}}},
\end{aligned} \tag{13}$$

where the last line applies exclusively to pure states. Using Eq. (11) with $\alpha = 1$, $K = \exp[-i(H - \langle\chi|H|\chi\rangle)t/\hbar - (\Gamma - i\langle\chi|\Gamma|\chi\rangle)t/\hbar]$, one finds $1/\sqrt{2}$ times the same expression.

From quantum speed limits to metrological bounds The ultimate bound to parameter estimation is dictated by the ability to efficiently discriminate neighboring quantum states. In a seminal paper [26], Braunstein and Caves (BC) derived a quantum Cramer-Rao bound for the uncertainty in the (local) estimation of a classical parameter ϕ of the form:

$$\Delta\phi \geq \frac{1}{\sqrt{\nu F_Q(\phi)}}, \tag{14}$$

where F_Q denotes the quantum Fisher information and ν is the total number of repetitions of the experiment where a ϕ-dependence is linearly imprinted via a general evolution. When the dynamics is unitary, an initial preparation of a probe state in a *cat* (GHZ) state of N subsystems allows to saturate the lower bound and achieve a Heisenberg-limited resolution where $\Delta\phi \sim 1/N$. If the N subsystems are used independently, so that the input state is factorizable as N product states, only the standard scaling dictated by the central limit theorem $\Delta\phi \sim 1/\sqrt{N}$ is achievable. This implies that the error bars in the actual estimation of a parameter ϕ could be reduced by $1\sqrt{N}$ by means of employing an entangled input probe provided that the system evolves unitarily. Whether or not the standard scaling can be surpassed when the system's dynamics dynamics is open is a most relevant issue where only partial results are known. Motivated by experiments on precision spectroscopy, where a phase difference is estimated which is proportional to the detuning between an external oscillator and a selected atomic frequency, we will focus here on phase estimation problems under dephasing noise. Assuming decoherence to be Markovian and affecting each subsystem independently (local noise assumption), it was shown in [17] that this type of noise renders product and maximally entangled states metrologically equivalent, and argued that Marko-

vian dephasing would restore the standard scaling with an optimal resolution to be achieved by a type of partially entangled states so that $\Delta\phi^{opt}/\Delta\phi^{p} = 1/\sqrt{e}$. Subsequent work proved this bound to be achievable asymptotically [27] but only very recently it was proved in all generality that the bound is sharp and coincides with the one imposed by the maximization of the quantum Fisher information [28]. The metrological equivalence of product and maximally entangled state preparations under Markovian decoherence can be predicted with the new bound eq. (8), which yields the ratio $t_{\rm GHZ} = t_p/N$, where $t_{\rm GHZ}$ and t_p are the optimal interrogation times when using maximally entangled and product state inputs, respectively. This can be easily shown by writing the dephasing master equation in the interaction picture as

$$\dot{\rho} = -\gamma\rho + \gamma\sigma_z\rho\sigma_z, \tag{15}$$

and considering a pure state $\rho_0 = |\chi\rangle\langle\chi|$ of the form $|\chi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Then,

$$\mathscr{L}^\dagger\rho_0 \;=\; -\gamma\rho_0 + \gamma\sigma_z\rho_0\sigma_z \tag{16}$$

and

$$\mathrm{tr}[(\mathscr{L}^\dagger\rho_0)^2] = 2\gamma^2. \tag{17}$$

This yields a minimal orthogonalization time $t_p = 1/\sqrt{2}\gamma$. Repeating the same procedure for a maximally entangled input of the (GHZ) form $\rho_0 = |\chi\rangle\langle\chi|$ with $|\chi\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$, we obtain an optimal interrogation time $t_{\rm GHZ} = t_p/N$ which leads to $\Delta\phi^{\rm GHZ} = \Delta\phi^{p}$ when the resolution is estimated operationally as $\Delta\phi = <\Delta O^2>^2 / \sqrt{\nu} \mid \frac{\partial <O>}{\partial\phi} \mid$, with O denoting a projective population measurement, which is known to be optimal for this specific context. Alternatively, we can estimate the Fisher information in the form

$$F(\rho_\phi) = \sum_i \frac{1}{p_i}\left(\frac{\partial p_i}{\partial\phi}\right)^2, \tag{18}$$

where $p_i = \mathrm{tr}(\rho_\phi P_i)$ and P_i is a population projective measurement. Note that this measurement procedure is optimal in this context. The resulting expressions for product and cat states are, respectively

$$F_p \;=\; Ne^{-2\gamma t}t^2, \tag{19}$$

$$F_{\rm GHZ} \;=\; N^2 e^{-2N\gamma t}t^2. \tag{20}$$

The ratio $\Delta\phi^{\rm GHZ}/\Delta\phi^{p} = \sqrt{(\nu_p F_p)/(\nu_{ghz}F_{\rm GHZ})}$ therefore equals 1 when considering the optimal interrogations times as dictated by the bound eq. (8). Moreover, for pure states $\rho_0 = |\chi\rangle\langle\chi|$ and the case of Markovian pure dephasing $\mathscr{L}^\dagger\rho_0 = \gamma\sum_k[-\rho_0 + \sigma_z^{(k)}\rho_0\sigma_z^{(k)}]$ we have that $\sqrt{\mathrm{tr}[(\mathscr{L}^\dagger\rho_0)^2]} = \gamma\sqrt{(N^2 - 2N\sum_k|\langle\chi|\sigma_z^{(k)}|\chi\rangle|^2 + \sum_{kl}|\langle\chi|\sigma_z^{(k)}\sigma_z^{(l)}|\chi\rangle|^2} \leq \sqrt{2}\gamma N$ (Note that this may be generalized to the mixed state case and any form of *local* noise as the locality implies that number of terms in $\mathscr{L}^\dagger\rho_0$ grows linearly in the number of subsystems N). Then with eq. (8) and the fact that the Fisher information obeys $F \leq N^2$ [29, 30], the limit on the speed of evolution imposes the persistence of the standard scaling $\Delta\phi \sim 1/\sqrt{N}$ no matter how weak the dephasing rate. This is a result that is now firmly established [28, 31] and that comes out in a rather natural fashion within this new framework.

So far we have exploited specifically the fact that the system's dynamics is ruled by a Lindblad master equation. However, our initial derivation considered a dynamical map that is CPT but not necessarily divisible. As a result, the bound could be valid for non Markovian dynamics that admit a representation in terms of a CP map. We have evaluated the prediction for the optimal interrogation times of product and cat states for a model of non-Markovian dephasing of this type, as proposed in [32], and obtained the ratio $t_{\rm GHZ} = t_p/N$, just as in the Markov case. This seems to be in contradiction with recent results for models of non Markovian dephasing, which predict a ratio $t_{\rm GHZ} = t_p/\sqrt{N}$ and raises an interesting conjecture with which we finish this section. There could exist forms of coloured noise for which the metrological equivalence between cats-products input probes still holds. This inequivalence in the achievable resolution of a phase estimation could then be exploited to quantitatively quantify non-Markovianity.

Conclusions– A bound to the speed of evolution under an open-system dynamics has been provided, generalising the classic result by Mandelstam and Tamm known for the unitary case. In the Markovian limit, we have shown that the adjoint of the generator of the dynamical semigroup plays the role of the commutator with the Hamiltonian in the MT bound. Despite the fact that the bound is not tight, in the sense of non coinciding with the unitary solution for closed systems, it allows to naturally predict the inaccessibility of the Heisenberg limit under Markovian noise. Moreover, when using the general form of the bound for universal channels, the new limit on the speed of evolution suggest the inequivalence of different forms of coloured noise for precision spectroscopy. Our results are applicable to a wide variety of scenarios including bounding decoherence rates [33], and quantum speed limits in dissipative state preparation [34], quantum computation and simulation assisted by dissipation [35].

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Derivation of the bound using quantum channels

Any kind of time evolution of a quantum state ρ_0 can be written in the form $\rho_t = \sum_\alpha K_\alpha(t,0)\rho_0 K_\alpha^\dagger(t,0)$. In particular, $K_\alpha(t,0)$ is independent of ρ_0 if the dynamical map is induced from an extended system with the initial contidion $\rho_0^{SE} = \rho_0 \otimes \rho_0^E$. Then, dynamical map is said to be universal. Let such map govern the evolution and consider

$$f(t) = \frac{\mathrm{tr}[\rho_0\rho_t]}{\mathrm{tr}\rho_0^2} = \frac{1}{\mathrm{tr}\rho_0^2}\sum_\alpha \mathrm{tr}[\rho_0 K_\alpha(t,0)\rho_0 K_\alpha^\dagger(t,0)]. \tag{21}$$

For compactness, let us denote $K_\alpha = K_\alpha(t,0)$. It follows that

$$\begin{aligned} \dot f(t) &= \frac{\mathrm{tr}[\rho_0 d_t\rho_t]}{\mathrm{tr}\rho_0^2}, \\ &= \frac{1}{\mathrm{tr}\rho_0^2}\sum_\alpha \mathrm{tr}[\rho_0(\dot K_\alpha\rho_0 K_\alpha^\dagger + K_\alpha\rho_0\dot K_\alpha^\dagger]. \end{aligned} \tag{22}$$

Taking the absolute value at both sides,

$$|\dot f(t)| \le \frac{1}{\sqrt{\mathrm{tr}[\rho_0^2]}}\sum_\alpha \left[|\mathrm{tr}(\rho_0\dot K_\alpha\rho_0 K_\alpha^\dagger)| + |\mathrm{tr}(\rho_0 K_\alpha\rho_0\dot K_\alpha^\dagger)|\right]. \tag{23}$$

Using the Cauchy-Schwarz inequality $|\mathrm{tr}(AB)| \le [\mathrm{tr}(A^\dagger A)\mathrm{tr}(B^\dagger B)]^{\frac{1}{2}}$,

$$\begin{aligned} |\dot f(t)| &\le \frac{1}{\sqrt{\mathrm{tr}[\rho_0^2]}}\sum_\alpha\Big(\sqrt{\mathrm{tr}[\dot K_\alpha\rho_0 K_\alpha^\dagger K_\alpha\rho_0\dot K_\alpha^\dagger]} \\ &\quad + \sqrt{\mathrm{tr}[K_\alpha\rho_0\dot K_\alpha^\dagger\dot K_\alpha\rho_0 K_\alpha^\dagger]}\Big), \\ &= \frac{2}{\sqrt{\mathrm{tr}[\rho_0^2]}}\sum_\alpha\sqrt{\mathrm{tr}[\rho_0 K_\alpha^\dagger K_\alpha\rho_0\dot K_\alpha^\dagger\dot K_\alpha]}. \end{aligned} \tag{24}$$

We next use the fact that $|\int dt \dot f(t)| \le \int dt |\dot f(t)|$. Parametrizing $f(t) = \cos\theta$, a bound can be derived

$$\begin{aligned} \tau &\ge \frac{|\cos\theta - 1|}{2}\frac{\sqrt{\mathrm{tr}[\rho_0^2]}}{\overline{\overline{\sum_\alpha\sqrt{\mathrm{tr}[\rho_0 K_\alpha^\dagger K_\alpha\rho_0\dot K_\alpha^\dagger\dot K_\alpha]}}}}, \\ &\ge \frac{2\theta^2}{\pi^2}\frac{\sqrt{\mathrm{tr}[\rho_0^2]}}{\overline{\overline{\sum_\alpha\sqrt{\mathrm{tr}[\rho_0 K_\alpha^\dagger K_\alpha\rho_0\dot K_\alpha^\dagger\dot K_\alpha]}}}}, \\ &= \frac{2\theta^2}{\pi^2}\frac{\sqrt{\mathrm{tr}[\rho_0^2]}}{\overline{\overline{\sum_\alpha||K_\alpha\rho_0\dot K_\alpha^\dagger||}}}, \end{aligned} \tag{25}$$

where $\overline{X} = \tau_\theta^{-1}\int_0^{\tau_\theta} X dt$ and $||A|| = \sqrt{\mathrm{tr}(A^\dagger A)}$ is the Hilbert-Schmidt norm of A. In the second line, we have used $|\cos\theta - 1| \ge 4\theta^2/\pi^2$.